Physics 606 Exam 2 Skeleton Solution

1. (a) $\boxed{[AB,C]_{-}} = ABC - CAB$

$= ABC \mp ACB \pm ACB - CAB$

$= A(BC \mp CB) \pm (AC \mp CA)B$

$= \boxed{A[B,C]_{\mp} \pm [A,C]_{\mp} B}$

(b) $i\hbar \dfrac{da}{dt} = [a,H]$

$= [a, a^{+}a]\hbar\omega$

$= -[a^{+}a, a]\hbar\omega$

$= -\left(a^{+}\underbrace{[a,a]_{\mp}}_{=0} \pm \underbrace{\underbrace{[a^{+},a]_{\mp}}_{=\mp[a,a^{+}]}\, a}_{=-1}\right)\hbar\omega$

$= +\hbar\omega a$

$\Rightarrow \boxed{\dfrac{da}{dt} = -i\omega a}$ and similarly $\boxed{\dfrac{db}{dt} = -i\omega b}$

(c) $\dfrac{da}{a} = -i\omega dt \Rightarrow \ln a = -i\omega t + \text{const} \Rightarrow a = \text{const} \times e^{-i\omega t}$

$\Rightarrow \boxed{a = a(0)e^{-i\omega t}}$

similarly $\boxed{b = b(0)e^{-i\omega t}}$

2. Using Hilbert space description:

(a) $L_{\pm}|\ell m\rangle = \sqrt{\ell(\ell+1) - m(m\pm 1)}\,\hbar\,|\ell, m\pm 1\rangle$

$L_{\pm} = L_x \pm i L_y \Rightarrow L_x = \frac{1}{2}(L_{+} + L_{-})$

$L_{+}|\Psi\rangle = \cancel{a\sqrt{2-2}} + b\sqrt{2-0}\,\hbar\,|1,1\rangle + c\sqrt{2-0}\,\hbar\,|1,0\rangle$

$L_{-}|\Psi\rangle = a\sqrt{2-0}\,\hbar\,|1,0\rangle + b\sqrt{2-0}\,\hbar\,|1,-1\rangle + \cancel{c\sqrt{2-2}}$

$\langle L_{+}\rangle = \langle\Psi|L_{+}|\Psi\rangle = \sqrt{2}\,\hbar\,(a^{*}b + b^{*}c)$

$\langle L_{-}\rangle = \langle\Psi|L_{-}|\Psi\rangle = \sqrt{2}\,\hbar\,(b^{*}a + c^{*}b)$

$\boxed{\langle L_x\rangle =} \frac{1}{2}(\langle L_{+}\rangle + \langle L_{-}\rangle) = \boxed{\frac{1}{\sqrt{2}}\hbar\,(a^{*}b + b^{*}a + b^{*}c + c^{*}b)}$

(b) $\boxed{\langle L^2\rangle =} 1(1+1)\hbar^2 = \boxed{2\hbar^2}$ since $L^2|\Psi\rangle = \ell(\ell+1)\hbar^2|\Psi\rangle$ with $\ell = 1$

[continued]

2. (c) $L_x|\Psi\rangle = \frac{1}{2}(L_+ + L_-)|\Psi\rangle$

$$= \frac{1}{\sqrt{2}}\hbar\left(b\,|1,1\rangle + (c+a)|1,0\rangle + b\,|1,-1\rangle\right)$$

so $\frac{1}{\sqrt{2}}\hbar b = a\hbar$, $\frac{1}{\sqrt{2}}\hbar(c+a) = b\hbar$, $\frac{1}{2}\hbar b = c\hbar$

or $b = \sqrt{2}\,a$, $c+a = \sqrt{2}\,b$, $b = \sqrt{2}\,c$

In general, $|\Psi\rangle = \frac{b}{|b|}\left(\frac{1}{2}|1,1\rangle + \frac{1}{\sqrt{2}}|1,0\rangle + \frac{1}{2}|1,-1\rangle\right)$

but one choice is $b = \frac{1}{\sqrt{2}}$.

3. (a) $L_z = x p_y - y p_x$

[this is one way, but all components can be done at once with ε_{ijk}]

$$i\hbar \frac{dL_z}{dt} = [L_z, H]$$

Again, this is just one way.

$$= [x p_y, H] - [y p_x, H]$$

$$= x\underbrace{[p_y, H]}_{= -i\hbar\frac{\partial H}{\partial y} = -i\hbar\frac{\partial V}{\partial y}} + \underbrace{[x, H]}_{= i\hbar\frac{\partial H}{\partial p_x} = i\hbar\frac{p_x}{m}} p_y - y\underbrace{[p_x, H]}_{= i\hbar\frac{\partial V}{\partial x}} - \underbrace{[y, H]}_{= i\hbar\frac{p_y}{m}} p_x$$

$$\Rightarrow \frac{dL_z}{dt} = -\left(x\frac{\partial V}{\partial y} - y\frac{\partial V}{\partial x}\right) \quad \text{since } p_x p_y = p_y p_x$$

$$= -\left(\vec{r}\times\vec{\nabla}V\right)_z$$

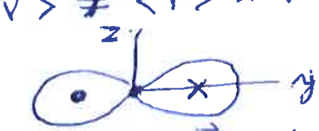
Similarly for $\frac{dL_y}{dt}$ & $\frac{dL_x}{dt}$:

$$\boxed{\frac{d\vec{L}}{dt} = -(\vec{r}\times\vec{\nabla}V)} = \vec{r}\times\vec{F}, \quad \vec{F} = -\vec{\nabla}V$$

[Yet again, there are various correct ways of getting this result.]

(b) There are many possible counterexamples (or demonstrations that $\langle\vec{r}\times\vec{\nabla}V\rangle \neq \langle\vec{r}\rangle\times\vec{\nabla}V(\langle\vec{r}\rangle)$;

E.g., for wavefunction shown,



$\langle\vec{r}\rangle = 0$ and $F(\langle\vec{r}\rangle) = 0$

$\vec{F}$ out of paper $\quad$ $\vec{F}$ into paper

but $\langle\vec{r}\times\vec{F}\rangle \neq 0$ is in z direction

4. Let us use Hilbert space description again:

(a) $\boxed{E =} \langle \psi | H | \psi \rangle$ with $H\psi_{n\ell m} = E_{n\ell m}\psi_{n\ell m}$

$$= \frac{1}{10}\left(2^2 E_{1,0,0} + E_{2,1,0} + (\sqrt{2})^2 E_{2,1,1} + (\sqrt{3})^2 E_{2,1,-1}\right)$$

$$= \frac{1}{10}(4E_1 + 6E_2) \quad \text{since } E_{n\ell m} \text{ is really } E_n$$

$$= \frac{1}{10}\left[4(-13.6\,\text{eV}) + 6(-13.6\,\text{eV})/2^2\right]$$

$$= \boxed{-7.47\ \text{eV}}$$

(b) $|\psi(t)\rangle = e^{-iHt/\hbar}|\psi(0)\rangle$

$\Rightarrow$ amplitude for state $|n11\rangle$ is

$$\langle n11|\psi(t)\rangle = \langle n11|e^{-iHt/\hbar}|\psi(0)\rangle$$

$$= \langle n11|e^{-iE_{n11}t/\hbar}|\psi(0)\rangle$$

[since $\langle i|e^{-iHt/\hbar} = \langle i|e^{-iE_i t/\hbar}$]

$$= e^{-iE_{n11}t/\hbar}\underbrace{\langle n11|\psi(0)\rangle}_{=\delta_{n2}\frac{\sqrt{2}}{\sqrt{10}}}$$

$$= e^{-iE_{n11}t/\hbar}\,\delta_{n2}\frac{1}{\sqrt{5}}$$

$\Rightarrow$ probability of state $|n11\rangle$ is

$$\boxed{|\langle n11|\psi(t)\rangle|^2 = \frac{1}{5}\delta_{n2}} \qquad [\text{since } |e^{-ix}|^2 = 1]$$

[The other way is to obtain $e^{-iHt/\hbar}|\psi(0)\rangle$.]